

Momentum relation and classical limit in the future-not-included complex action theory

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Abstract

Studying the time development of the expectation value in the future-not-included complex action theory we point out that the momentum relation (relation analogous to $p = \frac{\partial L}{\partial \dot{q}}$), which was derived via Feynman path integral and was shown to be right in the future-included theory in our previous papers, is not valid in the future-not-included theory. We provide the correct momentum relation in the future-not-included theory, and argue that the future-not-included classical theory is described by a certain real action. In addition we provide another way to understand the time development of the future-not-included theory by utilizing the future-included theory. Furthermore, applying the method used in our previous paper to the future-not-included theory properly by introducing a formal Lagrangian, we derive the correct momentum relation in the future-not-included theory.

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§1. Introduction

Complex action theory (CAT) is one of the trials to extend quantum theories by allowing its action to be complex. With the expectation that the imaginary part of the action would give some falsifiable predictions, the CAT has been studied recently.^{1),2)} So far various interesting suggestions have been made for Higgs mass,³⁾ quantum mechanical philosophy,^{4),5),6)} some fine-tuning problems,^{7),8)} black holes,⁹⁾ De Broglie-Bohm particle and a cut-off in loop diagrams.¹⁰⁾ Related to the CAT, integration contours in the complex plane^{11),12)} complex Langevin equations¹³⁾ and complexified solution set¹⁴⁾¹⁵⁾ have been also studied.

In ref.¹⁶⁾ in a system with a non-Hermitian diagonalizable bounded Hamiltonian \hat{H} , introducing a proper inner product^{*)} and considering the long time development of some state, we effectively obtained a Hermitian Hamiltonian. We note that \hat{H} is generically non-Hermitian, so it does not belong to a class of the PT symmetric non-Hermitian Hamiltonians, which has been intensively studied recently.^{20),21),17),18),19)} For the details of the PT symmetric non-Hermitian Hamiltonians, see the reviews^{22),23),24),25)} and the references therein. In addition, non-Hermitian time-dependent Hamiltonians are studied in ref.²⁶⁾ In ref.²⁷⁾ introducing various mathematical tools such as a modified set of complex conjugate, real and imaginary parts, Hermitian conjugates and bras, complex delta function etc., we explicitly constructed non-Hermitian operators of coordinate and momentum, \hat{q}_{new} and \hat{p}_{new} , and the eigenstates of their Hermitian conjugates $|q\rangle_{new}$ and $|p\rangle_{new}$ for complex q and p by utilizing coherent states of harmonic oscillators. Indeed, $|q\rangle$, which obeys $\hat{q}|q\rangle = q|q\rangle$, is defined only for real q , i.e. the eigenvalue of the Hermitian \hat{q} , so q is not allowed to be complex unless \hat{q} is extended to a non-Hermitian operator. Only in our complex coordinate formalism can we deal with complex q and p . This formalism would be a part of proof of consistency in using complex q and p in contours of integration for WKB approximation etc. in the usual real action theory (RAT). In ref.²⁸⁾ using this formalism we explicitly examined the momentum and Hamiltonian in the CAT via the Feynman path integral (FPI). We studied the time development of some ξ -parametrized state, which is a solution to a kind of an eigenvalue problem of a momentum operator. Finding the value of ξ that gives the largest contribution in FPI, we derived the momentum relation $p = m\dot{q}$ and Hamiltonian.

The future-included theory, i.e. the theory including not only a past time but also a future time as an integration interval of time, was studied in ref.¹⁾ The authors of ref.¹⁾ introduced the future state $|B(T_B)\rangle$ at the final time $T_B = \infty$ as well as the past state $|A(T_A)\rangle$ at the initial time $T_A = -\infty$. The states $|A(T_A)\rangle$ and $|B(T_B)\rangle$ time-develop according to the non-Hermitian Hamiltonians \hat{H} and $\hat{H}_B = \hat{H}^\dagger$, respectively. The authors of ref.¹⁾

*) Similar inner products were studied also in refs.^{17),18),19)}

speculated a correspondence of the future-included theory to the future-not-included one, i.e. $\langle \hat{O} \rangle^{BA} \simeq \langle \hat{O} \rangle^{AA}$, where $\langle \hat{O} \rangle^{BA} \equiv \frac{\langle B(t) | \hat{O} | A(t) \rangle}{\langle B(t) | A(t) \rangle}$, $\langle \hat{O} \rangle^{AA} \equiv \frac{\langle A(t) | \hat{O} | A(t) \rangle}{\langle A(t) | A(t) \rangle}$, and t is the present time. In the RAT the matrix element $\langle \hat{O} \rangle^{BA}$ is called weak value²⁹⁾ and has been intensively studied. For the details of the weak value, see the reviews^{30), 31), 32), 33)} and the references therein. In refs.^{34), 35)} we investigated $\langle \hat{O} \rangle^{BA}$ carefully, and found that if we regard it as an expectation value, then we obtain the Heisenberg equation, the Ehrenfest's theorem and a conserved probability current density. This result strongly suggests that we can regard $\langle \hat{O} \rangle^{BA}$ as the expectation value in the future-included theory. Furthermore, using both the complex coordinate formalism²⁷⁾ and the automatic hermiticity mechanism,^{16), 27)} i.e., a mechanism to obtain a Hermitian Hamiltonian after a long time development, we obtained a correspondence principle that $\langle \hat{O} \rangle^{BA}$ for large $T_B - t$ and large $t - T_A$ is almost equivalent to $\langle \hat{O} \rangle_{Q'}^{AA}$ for large $t - T_A$, where Q' is a Hermitian operator which is used to define a proper inner product. Thus the future-included theory is not excluded though it looks exotic.

As for the momentum relation, in ref.³⁴⁾ we obtained $\langle \hat{p}_{new} \rangle^{BA} = m \frac{d}{dt} \langle \hat{q}_{new} \rangle^{BA}$ in the case of the future-included theory. This is consistent with the momentum relation $p = m\dot{q}$, which we derived via FPI in ref.²⁸⁾ But how about it in the future-not-included theory? In the future-not-included theory $\langle \hat{q}_{new} \rangle^{AA}$ and $\langle \hat{p}_{new} \rangle^{AA}$ are real, if we replace \hat{q}_{new} and \hat{p}_{new} with Hermitian \hat{q} and \hat{p} respectively. On the other hand, $m\dot{q}$ is complex because m is complex. Thus we encounter a contradiction. This is quite in contrast to the case of the future-included theory, where $\langle \hat{q}_{new} \rangle^{BA}$ and $\langle \hat{p}_{new} \rangle^{BA}$ are complex even if \hat{q}_{new} and \hat{p}_{new} are replaced with \hat{q} and \hat{p} respectively, so we do not have such a contradiction. This fact suggests that the momentum relation $p = m\dot{q}$ is not valid in the future-not-included theory.

Thus we are motivated to examine the momentum relation in the future-not-included theory. In this paper studying the time development of $\langle \hat{O} \rangle^{AA}$ we argue that the momentum relation in the future-not-included theory is not given by $p = m\dot{q}$ but by another expression $p = m_{\text{eff}}\dot{q}$, where m_{eff} is a certain real mass. Moreover, since the effect of the anti-Hermitian part of the Hamiltonian is suppressed in the classical limit, we claim that classical theory of the future-not-included theory is described by the real part of the non-Hermitian Hamiltonian or a certain real action S_{eff} . In addition, we present another way to understand the time development of the future-not-included theory by utilizing the future-included theory. Furthermore, we discuss how we can utilize the method studied in ref.²⁸⁾ to obtain the correct momentum relation in the future-not-included theory. In the method we analyze the time development of ξ -parametrized state in a transition amplitude from initial time to final time, where the present time t is supposed to be between the initial and final times. This is the case for the future-included theory, but not for the future-not-included theory. Therefore, to apply the method to the future-not-included theory properly, we introduce a

formal Lagrangian by rewriting the transition amplitude in the future-not-included theory $\langle A(t)|A(t)\rangle$ into such an expression as $\langle B(t)|A(t)\rangle$, which is the transition amplitude in the future-included theory. We argue that using this formal Lagrangian in the method we obtain $p = m_{\text{eff}}\dot{q}$, the correct momentum relation in the future-not-included theory.

This paper is organized as follows. In section 2 we review the complex coordinate formalism proposed in ref.²⁷⁾ In section 3 following ref.²⁸⁾ we explain the method to derive the momentum relation $p = m\dot{q}$ via Feynman path integral. In section 4 based on ref.³⁴⁾ we show that $\langle \hat{\mathcal{O}} \rangle^{BA}$ behaves as if an expectation value of some operator $\hat{\mathcal{O}}$ in the future-included theory. Also, we obtain the relation $\langle \hat{p}_{\text{new}} \rangle^{BA} = m \frac{d}{dt} \langle \hat{q}_{\text{new}} \rangle^{BA}$, which is consistent with the momentum relation obtained in ref.²⁸⁾ In section 5 studying $\langle O \rangle^{AA}$ we obtain the momentum relation in the future-not-included theory, $p = m_{\text{eff}}\dot{q}$. Moreover, we argue that the classical theory is described by a certain real action S_{eff} . Furthermore we provide another way to understand the time development of the future-not-included theory by making use of the future-included theory. In section 6 we apply the method of ref.²⁸⁾ to the future-not-included theory properly by introducing the formal Lagrangian, and derive the momentum relation in the future-not-included theory, which is consistent with that derived in section 5. Section 7 is devoted to discussion.

§2. Complex coordinate formalism

In this section we briefly review the complex coordinate formalism that we proposed in ref.²⁷⁾ so that we can deal with complex coordinate q and momentum p properly not only in the CAT but also in a real action theory (RAT), where we encounter them at the saddle point in WKB approximation, etc.

2.1. Non-Hermitian operators \hat{q}_{new} and \hat{p}_{new} , and the eigenstates of their Hermitian conjugates $|q\rangle_{\text{new}}$ and $|p\rangle_{\text{new}}$

We can construct the non-Hermitian operators of coordinate and momentum, \hat{q}_{new} and \hat{p}_{new} , and the eigenstates of their Hermitian conjugates $|q\rangle_{\text{new}}$ and $|p\rangle_{\text{new}}$ such that

$$\hat{q}_{\text{new}}^\dagger |q\rangle_{\text{new}} = q |q\rangle_{\text{new}}, \quad (2.1)$$

$$\hat{p}_{\text{new}}^\dagger |p\rangle_{\text{new}} = p |p\rangle_{\text{new}}, \quad (2.2)$$

$$[\hat{q}_{\text{new}}, \hat{p}_{\text{new}}] = i\hbar, \quad (2.3)$$

for complex q and p by formally utilizing two coherent states. Our proposal is to replace the usual Hermitian operators of coordinate and momentum \hat{q} , \hat{p} , and their eigenstates $|q\rangle$ and $|p\rangle$, which obey $\hat{q}|q\rangle = q|q\rangle$, $\hat{p}|p\rangle = p|p\rangle$, and $[\hat{q}, \hat{p}] = i\hbar$ for real q and p , with $\hat{q}_{\text{new}}^\dagger$, $\hat{p}_{\text{new}}^\dagger$,

$|q\rangle_{new}$ and $|p\rangle_{new}$. The explicit expressions for \hat{q}_{new} , \hat{p}_{new} , $|q\rangle_{new}$ and $|p\rangle_{new}$ are given by^{*)}

$$\hat{q}_{new} \equiv \frac{1}{\sqrt{1-\epsilon\epsilon'}} (\hat{q} - i\epsilon\hat{p}), \quad (2.4)$$

$$\hat{p}_{new} \equiv \frac{1}{\sqrt{1-\epsilon\epsilon'}} (\hat{p} + i\epsilon'\hat{q}), \quad (2.5)$$

$$|q\rangle_{new} \equiv \left(\frac{1-\epsilon\epsilon'}{4\pi\hbar\epsilon}\right)^{\frac{1}{4}} e^{-\frac{1}{4\hbar\epsilon}(1-\epsilon\epsilon')q^2} \left|\sqrt{\frac{1-\epsilon\epsilon'}{2\hbar\epsilon}}q\right\rangle_{coh}, \quad (2.6)$$

$$|p\rangle_{new} \equiv \left(\frac{1-\epsilon\epsilon'}{4\pi\hbar\epsilon'}\right)^{\frac{1}{4}} e^{-\frac{1}{4\hbar\epsilon'}(1-\epsilon\epsilon')p^2} \left|i\sqrt{\frac{1-\epsilon\epsilon'}{2\hbar\epsilon'}}p\right\rangle_{coh'}, \quad (2.7)$$

where $|\lambda\rangle_{coh}$ is a coherent state parametrized with a complex parameter λ defined up to a normalization factor by $|\lambda\rangle_{coh} \equiv e^{\lambda a^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}|n\rangle$, and this satisfies the relation $a|\lambda\rangle_{coh} = \lambda|\lambda\rangle_{coh}$. Here $a = \sqrt{\frac{1}{2\hbar\epsilon}}(\hat{q} + i\epsilon\hat{p})$ and $a^\dagger = \sqrt{\frac{1}{2\hbar\epsilon}}(\hat{q} - i\epsilon\hat{p})$ are annihilation and creation operators. In eq.(2.7) $|\lambda\rangle_{coh'} \equiv e^{\lambda a'^\dagger}|0\rangle$, where a'^\dagger is given by $a'^\dagger = \sqrt{\frac{\epsilon'}{2\hbar}}(\hat{q} - i\frac{\hat{p}}{\epsilon'})$, is another coherent state defined similarly. Before seeing the properties of \hat{q}_{new} , \hat{p}_{new} , $|q\rangle_{new}$, and $|p\rangle_{new}$, we define a delta function of complex parameters in the next subsection.

2.2. The delta function

We define \mathcal{D} as a class of distributions depending on one complex variable $q \in \mathbf{C}$. Using a function $g : \mathbf{C} \rightarrow \mathbf{C}$ as a distribution^{**)} in the class \mathcal{D} , we define the following functional G

$$G[f] = \int_C f(q)g(q)dq \quad (2.8)$$

for any analytical function $f : \mathbf{C} \rightarrow \mathbf{C}$ with convergence requirements such that $f \rightarrow 0$ for $q \rightarrow \pm\infty$. The functional G is a linear mapping from the function f to a complex number. Since the simulated function g is supposed to be analytical in q , the path C , which is chosen to run from $-\infty$ to ∞ in the complex plane, can be deformed freely and so it is not relevant. As an example of such a distribution we could think of the delta function and approximate it by the smeared delta function defined for complex q by

$$g(q) = \delta_c^\epsilon(q) \equiv \sqrt{\frac{1}{4\pi\epsilon}} e^{-\frac{q^2}{4\epsilon}}, \quad (2.9)$$

^{*)} For simplicity we have replaced the parameters $m\omega$ and $m'\omega'$ used in ref.²⁷⁾ with $\frac{1}{\epsilon}$ and ϵ' .

^{**)} We recently noticed that another complex distribution was introduced in ref.³⁶⁾ It is different from ours in the following points: the complex distribution in ref.³⁶⁾ where $g(q)$ is supposed to have poles, is not well-defined by $g(q)$ alone, but needs the indication of which side of the poles C passes through. On the other hand, in our complex distribution we assume not the presence of poles of $g(q)$ but f not being a bounded entire function.

where ϵ is a finite small positive real number. For the limit of $\epsilon \rightarrow 0$ $g(q)$ converges in the distribution sense for complex q obeying the condition

$$L(q) \equiv (\operatorname{Re}(q))^2 - (\operatorname{Im}(q))^2 > 0. \quad (2.10)$$

For any analytical test function $f(q)^{*)}$ and any complex q_0 this $\delta_c^\epsilon(q)$ satisfies

$$\int_C f(q) \delta_c^\epsilon(q - q_0) dq = f(q_0), \quad (2.11)$$

as long as we choose the path C such that it runs from $-\infty$ to ∞ in the complex plane and at any q its tangent line and a horizontal line form an angle θ whose absolute value is within $\frac{\pi}{4}$ to satisfy the inequality (2.10). An example of permitted paths is shown in fig.1, and the domain of the delta function is drawn in fig.2.

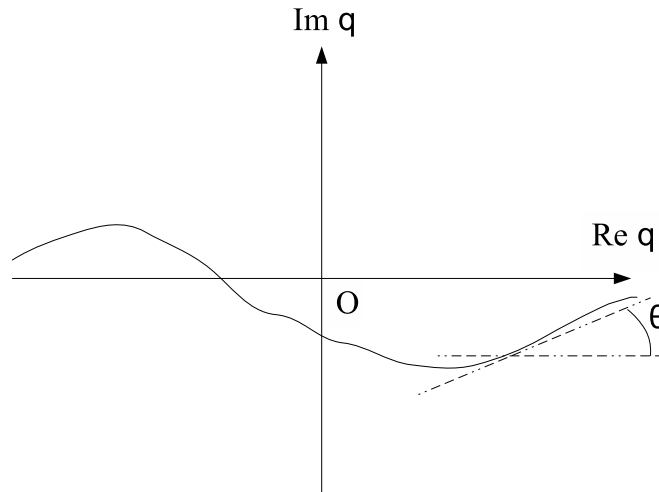


Fig. 1. An example of permitted paths

Next we extend the delta function to complex ϵ , and consider

$$\delta_c^\epsilon(aq) = \sqrt{\frac{1}{4\pi\epsilon}} e^{-\frac{1}{4\epsilon} a^2 q^2} \quad (2.12)$$

^{*)} Because of the Liouville theorem if f is a bounded entire function, f is constant. So we are considering as f an unbounded entire function or a function that is not entire but is holomorphic at least in the region on which the path runs.

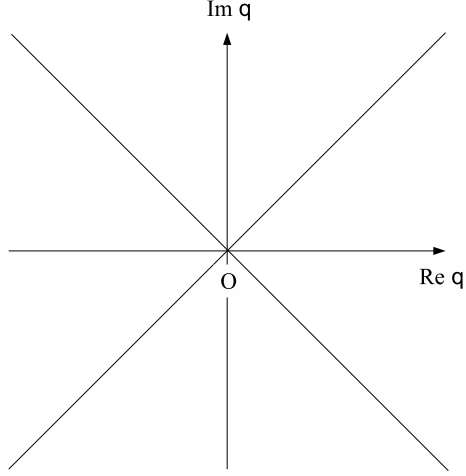


Fig. 2. Domain of the delta function

for non-zero complex a . We express ϵ , q , and a as $\epsilon = r_\epsilon e^{i\theta_\epsilon}$, $q = r e^{i\theta}$, and $a = r_a e^{i\theta_a}$. The convergence condition of $\delta_c^\epsilon(aq)$: $\text{Re}\left(\frac{a^2 q^2}{\epsilon}\right) > 0$ is expressed as

$$-\frac{\pi}{4} + \frac{1}{2}(\theta_\epsilon - 2\theta_a) < \theta < \frac{\pi}{4} + \frac{1}{2}(\theta_\epsilon - 2\theta_a), \quad (2.13)$$

$$\frac{3}{4}\pi + \frac{1}{2}(\theta_\epsilon - 2\theta_a) < \theta < \frac{5}{4}\pi + \frac{1}{2}(\theta_\epsilon - 2\theta_a). \quad (2.14)$$

For q , ϵ , and a such that eqs.(2.13)(2.14) are satisfied, $\delta_c^\epsilon(aq)$ behaves well as a delta function of aq , and we obtain the relation

$$\delta_c^\epsilon(aq) = \frac{\text{sign}(\text{Re}a)}{a} \delta_c^{\frac{\epsilon}{a^2}}(q), \quad (2.15)$$

where we have introduced an expression

$$\text{sign}(\text{Re}a) \equiv \begin{cases} 1 & \text{for } \text{Re}a > 0, \\ -1 & \text{for } \text{Re}a < 0. \end{cases} \quad (2.16)$$

2.3. New devices to handle complex parameters

To keep the analyticity in dynamical variables of FPI such as q and p , we define a modified set of a complex conjugate, real and imaginary parts, bras and Hermitian conjugates.

2.3.1. Modified complex conjugate $*_{\{\}}$

We define a modified complex conjugate for a function of n -parameters $f(\{a_i\}_{i=1,\dots,n})$ by

$$f(\{a_i\}_{i=1,\dots,n})^{*\{a_i|i \in A\}} = f^*(\{a_i\}_{i \in A}, \{a_i^*\}_{i \notin A}), \quad (2.17)$$

where A denotes the set of indices attached to the parameters in which we keep the analyticity, and $*$ on f acts on the coefficients included in f . For example, the complex conjugate $*_{q,p}$ of a function $f(q, p) = aq^2 + bp^2$ is written as $f(q, p)^{*_{q,p}} = a^*q^2 + b^*p^2$. The analyticity is kept in both q and p . For simplicity we express the modified complex conjugate as $*_{\{\}}$, where $\{\}$ is a symbolical expression for a set of parameters in which we keep the analyticity.

2.3.2. Modified real and imaginary parts $\text{Re}_{\{\}}$, $\text{Im}_{\{\}}$

We define the modified real and imaginary parts by using $*_{\{\}}$. We decompose some complex function f as

$$f = \text{Re}_{\{\}}f + i\text{Im}_{\{\}}f, \quad (2.18)$$

where $\text{Re}_{\{\}}f$ and $\text{Im}_{\{\}}f$ are the “ $\{\}$ -real” and “ $\{\}$ -imaginary” parts of f defined by

$$\text{Re}_{\{\}}f \equiv \frac{f + f^{*\{\}}}{2}, \quad (2.19)$$

$$\text{Im}_{\{\}}f \equiv \frac{f - f^{*\{\}}}{2i}. \quad (2.20)$$

For example, for $f = kq^2$, q -real and q -imaginary parts of f are expressed as $\text{Re}_q f = \text{Re}(k)q^2$ and $\text{Im}_q f = \text{Im}(k)q^2$, respectively. Especially, if f satisfies $f^{*\{\}} = f$, we say f is $\{\}$ -real, while if f obeys $f^{*\{\}} = -f$, we call r purely $\{\}$ -imaginary.

2.3.3. Modified bras ${}_m\langle |$ and $\langle |_{\{\}}$, and modified Hermitian conjugate $\dagger_{\{\}}$

For some state $|\lambda\rangle$ with some complex parameter λ , we define a modified bra ${}_m\langle \lambda|$ by

$${}_m\langle \lambda| \equiv \langle \lambda^*| \quad (2.21)$$

so that it preserves the analyticity in λ . In the special case of λ being real it becomes a usual bra. In addition we define a slightly generalized modified bra $\langle |_{\{\}}$ and a modified Hermitian conjugate $\dagger_{\{\}}$ of a ket. For example, ${}_{u,v}\langle u| = {}_u\langle u| = {}_m\langle u|$, $(|u\rangle)^{\dagger_{u,v}} = (|u\rangle)^{\dagger_u} = {}_m\langle u|$. We express the Hermitian conjugate $\dagger_{\{\}}$ of a ket symbolically as $(| \rangle)^{\dagger_{\{\}}} = \langle |_{\{\}}$. Also, we write the Hermitian conjugate $\dagger_{\{\}}$ of a bra as $(\langle |_{\{\}})^{\dagger_{\{\}}} = | \rangle$. Hence for a matrix element we have the relation $\langle u|A|v\rangle^{*\{\}} = \langle v|A^{\dagger}|u\rangle$.

2.4. Properties of \hat{q}_{new} , \hat{p}_{new} , $|q\rangle_{new}$ and $|p\rangle_{new}$, and a theorem for matrix elements

The states $|q\rangle_{new}$ and $|p\rangle_{new}$ are normalized so that they satisfy the following relations,

$${}_m\langle_{new} q'|q\rangle_{new} = \delta_c^{\epsilon_1}(q' - q), \quad (2.22)$$

$${}_m\langle_{new} p'|p\rangle_{new} = \delta_c^{\epsilon'_1}(p' - p), \quad (2.23)$$

where $\epsilon_1 = \frac{\hbar\epsilon}{1-\epsilon\epsilon'}$ and $\epsilon'_1 = \frac{\hbar\epsilon'}{1-\epsilon\epsilon'}$. We take ϵ and ϵ' sufficiently small, for which the delta functions converge for complex q , q' , p , and p' satisfying the conditions $L(q - q') > 0$ and $L(p - p') > 0$, where L is given in eq.(2.10). These conditions are satisfied only when q and q' or p and p' are on the same paths respectively. Eqs.(2.22)(2.23) represent the orthogonality relations for $|q\rangle_{new}$ and $|p\rangle_{new}$, and we have the following relations for complex q and p :

$$\int_C dq |q\rangle_{new} {}_m\langle_{new} q| = 1, \quad (2.24)$$

$$\int_C dp |p\rangle_{new} {}_m\langle_{new} p| = 1, \quad (2.25)$$

$$\hat{p}_{new}^\dagger |q\rangle_{new} = i\hbar \frac{\partial}{\partial q} |q\rangle_{new}, \quad (2.26)$$

$$\hat{q}_{new}^\dagger |p\rangle_{new} = \frac{\hbar}{i} \frac{\partial}{\partial p} |p\rangle_{new}, \quad (2.27)$$

$${}_m\langle_{new} q | p \rangle_{new} = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar} pq\right). \quad (2.28)$$

Thus \hat{q}_{new}^\dagger , \hat{p}_{new}^\dagger , $|q\rangle_{new}$ and $|p\rangle_{new}$ with complex q and p obey the same relations that \hat{q} , \hat{p} , $|q\rangle$, and $|p\rangle$ with real q and p satisfy. In the limits of $\epsilon \rightarrow 0$ and $\epsilon' \rightarrow 0$ $\delta_c^{\epsilon_1}(q' - q)$, $\delta_c^{\epsilon'_1}(p' - p)$ and $\exp\left(\frac{i}{\hbar} pq\right)$ in eqs.(2.22)(2.23)(2.28) are well-defined as distributions of the type \mathcal{D} . For real q' and p' , $|q'\rangle_{new}$ and $|p'\rangle_{new}$ become $|q'\rangle$ and $|p'\rangle$ respectively. Also, for them, \hat{q}_{new}^\dagger and \hat{p}_{new}^\dagger behave like \hat{q} and \hat{p} respectively. In addition we have the following theorem.

Theorem 1: The matrix element ${}_m\langle_{new} q' \text{ or } p' | \mathcal{O}(\hat{q}_{new}, \hat{q}_{new}^\dagger, \hat{p}_{new}, \hat{p}_{new}^\dagger) | q'' \text{ or } p'' \rangle_{new}$, where \mathcal{O} is a Taylor-expandable function, can be evaluated as if inside \mathcal{O} we had the hermiticity conditions $\hat{q}_{new} \simeq \hat{q}_{new}^\dagger \simeq \hat{q}$ and $\hat{p}_{new} \simeq \hat{p}_{new}^\dagger \simeq \hat{p}$ for q' , q'' , p' , p'' such that the resulting quantities are well-defined in the sense of distribution.

This theorem is understood by noticing that such a matrix element can be expressed as the summation of the products of factors made of q' , p' , q'' , p'' or their differential operators and distributions. Thus we do not have to worry about the anti-Hermitian terms in \hat{q}_{new} , \hat{q}_{new}^\dagger , \hat{p}_{new} and \hat{p}_{new}^\dagger , provided that we are satisfied with the result in the distribution sense.

§3. The method to derive the momentum relation via Feynman path integral

We briefly explain how we derived the momentum relation in ref.²⁸⁾

3.1. The derivation of the momentum relation and the Hamiltonian

The Feynman path integral (FPI) in the complex action theory (CAT) is described with the following Lagrangian – a typical example for a system with a single degree of freedom –,

$$L(q(t), \dot{q}(t)) = \frac{1}{2}m\dot{q}^2 - V(q), \quad (3.1)$$

where $V(q) = \sum_{n=2}^{\infty} b_n q^n$ is a potential term. For our later convenience we decompose V and L as $V = V_R + iV_I$ and $L = L_R + iL_I$, where V_R , V_I , L_R and L_I are given by

$$V_R \equiv \text{Re}_q(V) = \sum_{n=2}^{\infty} \text{Re} b_n q^n, \quad (3.2)$$

$$V_I \equiv \text{Im}_q(V) = \sum_{n=2}^{\infty} \text{Im} b_n q^n, \quad (3.3)$$

$$L_R \equiv \text{Re}_q(L) = \frac{1}{2}m_R\dot{q}^2 - V_R(q), \quad (3.4)$$

$$L_I \equiv \text{Im}_q(L) = \frac{1}{2}m_I\dot{q}^2 - V_I(q). \quad (3.5)$$

Here Re_q and Im_q are introduced in eqs.(2.19)(2.20), and we have decomposed m into its real and imaginary parts as $m = m_R + im_I$.

We consider the functional integral $\int_C e^{\frac{i}{\hbar} \int L(q, \dot{q}) dt} \mathcal{D}q$ by discretizing the time direction and writing \dot{q} as $\dot{q} = \frac{q(t+dt) - q(t)}{dt}$, where dt is assumed to be a small quantity. Since we use the Schrödinger representation for wave functions, to avoid the confusion with the Heisenberg representation we introduce the notations $q_t \equiv q(t)$ and $q_{t+dt} \equiv q(t+dt)$, which we regard as independent variables. We suppose that the asymptotic values of dynamical variables such as q and p are on the real axis, while parameters such as m and b_n are complex in general. The path C denotes an arbitrary path running from $-\infty$ to ∞ in the complex plane, and we can deform it as long as the integrand keeps the analyticity in q and p . To prevent the kinetic term in the integrand from blowing up for $\dot{q} \rightarrow \pm\infty$ along real axis we impose the condition $m_I \geq 0$ on m .

In FPI the time development of some wave function ${}_m\langle_{new} q_t | \psi(t) \rangle$ at some time t to $t + dt$ is described by

$${}_m\langle_{new} q_{t+dt} | \psi(t+dt) \rangle = \int_C e^{\frac{i}{\hbar} \Delta t L(q, \dot{q})} {}_m\langle_{new} q_t | \psi(t) \rangle dq_t, \quad (3.6)$$

where $L(q, \dot{q})$ is given by eq.(3.1), and C is an arbitrary path running from $-\infty$ to ∞ in the complex plane. In ref.²⁸⁾ to derive the momentum relation $p = \frac{\partial L}{\partial \dot{q}}$, we considered some wave function ${}_m\langle_{new} q_t | \xi \rangle$ that obeys

$${}_m\langle_{new} q_t | \hat{p}_{new} | \xi \rangle = \frac{\hbar}{i} \frac{\partial}{\partial q_t} {}_m\langle_{new} q_t | \xi \rangle$$

$$= \frac{\partial L}{\partial \dot{q}} \left(q_t, \frac{\xi - q_t}{dt} \right) {}_m\langle_{new} q_t | \xi \rangle, \quad (3.7)$$

where ξ is any number. Since the set $\{|\xi\rangle\}$ is an approximately reasonable basis which has roughly completeness $1 \simeq \int_C d\xi |\xi\rangle {}_m\langle_{anti} \xi|$ and orthogonality ${}_m\langle_{anti} \xi | \xi'\rangle \simeq \delta_c(\xi - \xi')$, where ${}_m\langle_{anti} \xi|$ is a dual basis of $|\xi\rangle$, we can expand the wave function ${}_m\langle_{new} q_t | \psi(t)\rangle$ into a linear combination of ${}_m\langle_{new} q_t | \xi\rangle$ as

$$\begin{aligned} {}_m\langle_{new} q_t | \psi(t)\rangle &\simeq \int_C d\xi {}_m\langle_{new} q_t | \xi\rangle {}_m\langle_{anti} \xi | \psi(t)\rangle \\ &= \int_C d\xi {}_m\langle_{new} q_t | \psi(t)\rangle|_{\xi}. \end{aligned} \quad (3.8)$$

Then solving eq.(3.7) we obtain

$$\begin{aligned} {}_m\langle_{new} q_{t+dt} | \psi(t+dt)\rangle|_{\xi} &= \sqrt{\frac{2\pi\hbar dt}{m}} {}_m\langle_{anti} \xi | \psi(t)\rangle \exp \left[\frac{im}{2\hbar dt} (q_{t+dt}^2 - \xi^2) \right] \\ &\times \left\{ \delta_c(\xi - q_{t+dt}) - \sum_{n=2} \left(\frac{\hbar dt}{m} \right)^n (-i)^n \frac{idt}{\hbar} b_n \frac{\partial^n \delta_c(\xi - q_{t+dt})}{\partial \xi^n} \right\}. \end{aligned} \quad (3.9)$$

Since ${}_m\langle_{new} q_{t+dt} | \psi(t+dt)\rangle|_{\xi}$ is equal to the linear combination of $\delta_c(q_{t+dt} - \xi)$ and its derivative, only the component with $\xi = q_{t+dt}$ contributes to ${}_m\langle_{new} q_{t+dt} | \psi(t+dt)\rangle$. Thus we have obtained the momentum relation in the sense of eq.(3.7),

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}. \quad (3.10)$$

Furthermore, we can estimate the right-hand side of eq.(3.6) explicitly as follows,

$$\begin{aligned} {}_m\langle_{new} q_{t+dt} | \psi(t+dt)\rangle &= \int_C' d\xi \int_C dq_t e^{\frac{i}{\hbar} \Delta t L(q, \dot{q})} {}_m\langle_{new} q_t | \xi\rangle {}_m\langle_{anti} \xi | \psi(t)\rangle \\ &\simeq \sqrt{\frac{2\pi i \hbar dt}{m}} {}_m\langle_{new} q_{t+dt} | \exp \left(-\frac{i}{\hbar} \hat{H} dt \right) | \psi(t)\rangle, \end{aligned} \quad (3.11)$$

where \hat{H} is given by

$$\hat{H} = H(\hat{q}_{new}, \hat{p}_{new}) = \frac{1}{2m} (\hat{p}_{new})^2 + V(\hat{q}_{new}). \quad (3.12)$$

Thus starting from eq.(3.6) we have found that the Hamiltonian \hat{H} has the same form as that in the RAT.

3.2. The derivation of the Lagrangian and momentum relation

Following ref.²⁸⁾ we derive the Lagrangian and momentum relation. We analyze the transition amplitude from an initial state $|i\rangle$ at time t_i to a final state $|f\rangle$ at time t_f , which is written as

$$\begin{aligned} & \langle f | e^{-\frac{i}{\hbar} \hat{H}(t_f - t_i)} | i \rangle \\ &= \int_C dq_1 \cdots dq_N \langle f | q_N \rangle_{new} {}_m \langle_{new} q_N | e^{-\frac{i}{\hbar} \hat{H} \Delta t} | q_{N-1} \rangle_{new} {}_m \langle_{new} q_{N-1} | \cdots | q_2 \rangle_{new} \\ & \quad \times {}_m \langle_{new} q_2 | e^{-\frac{i}{\hbar} \hat{H} \Delta t} | q_1 \rangle_{new} {}_m \langle_{new} q_1 | i \rangle, \end{aligned} \quad (3.13)$$

where we have divided the time interval $t_f - t_i$ into $N - 1$ pieces whose interval is $\Delta t = \frac{t_f - t_i}{N-1}$, and defined \dot{q}_j by $\dot{q}_j \equiv \frac{q_{j+1} - q_j}{\Delta t}$. Then since ${}_m \langle_{new} q_{j+1} | e^{-\frac{i}{\hbar} \hat{H} \Delta t} | q_j \rangle_{new}$ is rewritten as,

$$\begin{aligned} {}_m \langle_{new} q_{j+1} | e^{-\frac{i}{\hbar} H(\hat{p}_{new}, \hat{q}_{new}) \Delta t} | q_j \rangle_{new} &= \int_C dp_j e^{-\frac{i}{\hbar} H(p_j, q_j) \Delta t} {}_m \langle_{new} q_{j+1} | p_j \rangle_{new} {}_m \langle_{new} p_j | q_j \rangle_{new} \\ &= \int_C \frac{dp_j}{2\pi\hbar} \exp \left[\frac{i}{\hbar} \Delta t L(p_j, q_j, \dot{q}_j) \right], \end{aligned} \quad (3.14)$$

where $L(p_j, q_j, \dot{q}_j)$ is given by

$$\begin{aligned} L(p_j, q_j, \dot{q}_j) &= p_j \dot{q}_j - H(p_j, q_j) \\ &= -\frac{1}{2m} (p_j - m\dot{q}_j)^2 + \frac{1}{2} m \dot{q}_j^2 - V(q_j), \end{aligned} \quad (3.15)$$

the transition amplitude $\langle f | e^{-\frac{i}{\hbar} \hat{H}(t_f - t_i)} | i \rangle$ is estimated as

$$\begin{aligned} & \langle f | e^{-\frac{i}{\hbar} \hat{H}(t_f - t_i)} | i \rangle \\ &= \int_C \frac{dp_1}{2\pi\hbar} \cdots \frac{dp_{N-1}}{2\pi\hbar} dq_1 \cdots dq_N \langle f | q_N \rangle_{new} {}_m \langle_{new} q_1 | i \rangle \exp \left[\frac{i}{\hbar} \sum_{j=1}^{N-1} \Delta t L(p_j, q_j, \dot{q}_j) \right] \\ &= \int_C \mathcal{D}p \mathcal{D}q \psi_f(q_f)^{*q_f} \psi_i(q_i) \exp \left[\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(p, q, \dot{q}) \right], \end{aligned} \quad (3.16)$$

where in the second equality we have introduced $q_i = q_1$ and $q_f = q_N$. We perform the following Gaussian integral around the saddle point $p_j = m\dot{q}_j$,

$$\begin{aligned} \int_C \frac{dp_j}{2\pi\hbar} \exp \left[\frac{i}{\hbar} \Delta t L(p_j, q_j, \dot{q}_j) \right] &= \int_C \frac{dp_j}{2\pi\hbar} \exp \left[\frac{i}{\hbar} \Delta t \left\{ -\frac{1}{2m} (p_j - m\dot{q}_j)^2 + \frac{1}{2} m \dot{q}_j^2 - V(q_j) \right\} \right] \\ &= \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \exp \left[\frac{i}{\hbar} \Delta t L(\dot{q}_j, q_j) \right], \end{aligned} \quad (3.17)$$

where $L(\dot{q}_j, q_j) = \frac{1}{2} m \dot{q}_j^2 - V(q_j)$. Thus we have obtained the momentum relation (3.10) and the Lagrangian (3.1).

§4. Properties of the future-included theory

4.1. Future-included theory

Improving the definition given in ref.¹⁾ in ref.³⁴⁾ based on the complex coordinate formalism²⁷⁾ we introduced $|A(t)\rangle$ and $|B(t)\rangle$ by

$$\psi_A(q) = {}_m\langle_{new} q|A(t)\rangle = \int_{\text{path}(t)=q} e^{\frac{i}{\hbar} S_{T_A \text{ to } t}(\text{path})} D\text{path}, \quad (4.1)$$

$$\psi_B(q)^{*q} = \langle B(t)|q\rangle_{new} = \int_{\text{path}(t)=q} e^{\frac{i}{\hbar} S_{t \text{ to } T_B}(\text{path})} D\text{path}, \quad (4.2)$$

where $\text{path}(t) = q$ means the boundary condition at the present time t , and T_A and T_B are taken $-\infty$ and ∞ respectively. $|A(t)\rangle$ and $|B(t)\rangle$ are supposed to time-develop according to

$$i\hbar \frac{d}{dt} |A(t)\rangle = \hat{H} |A(t)\rangle, \quad (4.3)$$

$$i\hbar \frac{d}{dt} |B(t)\rangle = \hat{H}_B |B(t)\rangle, \quad (4.4)$$

where $\hat{H}_B = \hat{H}^\dagger$.

The authors of ref.¹⁾ speculated that the following matrix element^{*)} of some operator $\hat{\mathcal{O}}$

$$\langle \hat{\mathcal{O}} \rangle^{BA} \equiv \frac{\langle B(t)|\hat{\mathcal{O}}|A(t)\rangle}{\langle B(t)|A(t)\rangle} \quad (4.5)$$

corresponds to the expectation value in the future-not-included theory,

$$\langle \hat{\mathcal{O}} \rangle^{AA} \equiv \frac{\langle A(t)|\hat{\mathcal{O}}|A(t)\rangle}{\langle A(t)|A(t)\rangle}, \quad (4.6)$$

i.e. $\langle \hat{\mathcal{O}} \rangle^{BA} \simeq \langle \hat{\mathcal{O}} \rangle^{AA}$. In ref.^{34),35)} we investigated $\langle \hat{\mathcal{O}} \rangle^{BA}$ carefully. Using both the complex coordinate formalism²⁷⁾ and the automatic hermiticity mechanism,^{16),27)} i.e., a mechanism to obtain the Hermitian Hamiltonian after a long time development, we obtained a correspondence principle that $\langle \hat{\mathcal{O}} \rangle^{BA}$ for large $T_B - t$ and large $t - T_A$ is almost equivalent to $\langle \hat{\mathcal{O}} \rangle_{Q'}^{AA}$ for large $t - T_A$, where Q' is a Hermitian operator which is used to define a proper inner product.^{**)}

We note that $\langle \hat{\mathcal{O}} \rangle^{BA}$ is not an expectation value but a matrix element in a usual sense. But in ref.³⁴⁾ we found that if we regard it as an expectation value in the future-included theory, then we obtain the Heisenberg equation, the Ehrenfest's theorem and a conserved probability current density. This result strongly suggests that we can regard $\langle \hat{\mathcal{O}} \rangle^{BA}$ as the expectation value in the future-included theory.

^{*)} In the RAT the matrix element $\langle \hat{\mathcal{O}} \rangle^{BA}$ is called weak value²⁹⁾ and has been intensively studied. For the details of the weak value, see the reviews^{30),31),32),33)} and the references therein.

^{**)} In this paper we are not involved in the proper inner product, which is defined by making the Hamiltonian normal, for simplicity because it does not have an essential role in the study of this paper.

4.2. Heisenberg equation and Ehrenfest's theorem

In ref.³⁴⁾ we defined the Heisenberg operator,

$$\hat{\mathcal{O}}_H^{fi}(t, t_{\text{ref}}) \equiv \exp\left(\frac{i}{\hbar}\hat{H}(t - t_{\text{ref}})\right) \hat{\mathcal{O}} \exp\left(-\frac{i}{\hbar}\hat{H}(t - t_{\text{ref}})\right), \quad (4.7)$$

where \hat{H} is given in eq.(3.12) and t_{ref} is some reference time chosen arbitrarily such that $T_A \leq t_{\text{ref}} \leq T_B$. This Heisenberg operator, which appears in the numerator of $\langle \hat{\mathcal{O}} \rangle^{BA}$ as $\langle B(t) | \hat{\mathcal{O}} | A(t) \rangle = \langle B(t_{\text{ref}}) | \hat{\mathcal{O}}_H^{fi}(t, t_{\text{ref}}) | A(t_{\text{ref}}) \rangle$, obeys the Heisenberg equation

$$\frac{d}{dt} \hat{\mathcal{O}}_H^{fi}(t, t_{\text{ref}}) = \frac{i}{\hbar} [\hat{H}, \hat{\mathcal{O}}_H^{fi}(t, t_{\text{ref}})]. \quad (4.8)$$

In addition, since $\langle \hat{\mathcal{O}} \rangle^{BA}$ obeys

$$\frac{d}{dt} \langle \hat{\mathcal{O}} \rangle^{BA} = \langle \frac{i}{\hbar} [\hat{H}, \hat{\mathcal{O}}] \rangle^{BA}, \quad (4.9)$$

we obtain

$$\frac{d}{dt} \langle \hat{q}_{\text{new}} \rangle^{BA} = \frac{1}{m} \langle \hat{p}_{\text{new}} \rangle^{BA}, \quad (4.10)$$

$$\frac{d}{dt} \langle \hat{p}_{\text{new}} \rangle^{BA} = -\langle V'(\hat{q}_{\text{new}}) \rangle^{BA}, \quad (4.11)$$

and the Ehrenfest's theorem, $m \frac{d^2}{dt^2} \langle \hat{q}_{\text{new}} \rangle^{BA} = -\langle V'(\hat{q}_{\text{new}}) \rangle^{BA}$. Thus $\langle \hat{\mathcal{O}} \rangle^{BA}$ provides the time development of the saddle point for $\exp(\frac{i}{\hbar}S)$. Since eq.(4.10) is consistent with eq.(3.10), eq.(3.10) is confirmed to be the momentum relation in the future-included theory.

§5. Properties of the future-not-included theory

5.1. Heisenberg equation and Schrödinger equation

Following refs.^{16),27)} we explain the time development of $\langle \hat{\mathcal{O}} \rangle^{AA}$ given in eq.(4.6). Introducing a normalized state $|A(t)\rangle_N$ by

$$|A(t)\rangle_N \equiv \frac{1}{\sqrt{\langle A(t) | A(t) \rangle}} |A(t)\rangle, \quad (5.1)$$

we express $\langle \hat{\mathcal{O}} \rangle^{AA}$ as

$$\begin{aligned} \langle \hat{\mathcal{O}} \rangle^{AA} &= {}_N \langle A(t) | \hat{\mathcal{O}} | A(t) \rangle_N \\ &= {}_N \langle A(t_0) | \hat{\mathcal{O}}_H^{fni}(t, t_0) | A(t_0) \rangle_N, \end{aligned} \quad (5.2)$$

where we have introduced the Heisenberg operator $\hat{\mathcal{O}}_H^{fni}(t, t_0)$ by

$$\hat{\mathcal{O}}_H^{fni}(t, t_0) \equiv \frac{\langle A(t_0) | A(t_0) \rangle}{\langle A(t) | A(t) \rangle} e^{\frac{i}{\hbar}\hat{H}^\dagger(t-t_0)} \hat{\mathcal{O}} e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}. \quad (5.3)$$

This operator $\hat{\mathcal{O}}_H^{fni}(t, t_0)$ obeys the slightly modified Heisenberg equation,

$$\begin{aligned} i\hbar \frac{d}{dt} \hat{\mathcal{O}}_H^{fni}(t, t_0) &= \hat{\mathcal{O}}_H^{fni}(t, t_0) \hat{H} - \hat{H}^\dagger \hat{\mathcal{O}}_H^{fni}(t, t_0) - 2_N \langle A(t) | \hat{H}_a | A(t) \rangle_N \hat{\mathcal{O}}_H^{fni}(t, t_0) \\ &= [\hat{\mathcal{O}}_H^{fni}(t, t_0), \hat{H}_h] + \left\{ \hat{\mathcal{O}}_H^{fni}(t, t_0), \hat{H}_a - {}_N \langle A(t) | \hat{H}_a | A(t) \rangle_N \right\}, \end{aligned} \quad (5.4)$$

where \hat{H}_h and \hat{H}_a are Hermitian and anti-Hermitian parts of \hat{H} respectively. We note that eq.(5.4) is more complicated than the Heisenberg equation in the future-included theory, eq.(4.8). In addition, $|A(t)\rangle_N$ obeys the slightly modified Schrödinger equation,

$$\begin{aligned} i\hbar \frac{d}{dt} |A(t)\rangle_N &= \hat{H} |A(t)\rangle_N - {}_N \langle A(t) | \hat{H}_a | A(t) \rangle_N |A(t)\rangle_N \\ &= \hat{H}_h |A(t)\rangle_N + \left(\hat{H}_a - {}_N \langle A(t) | \hat{H}_a | A(t) \rangle_N \right) |A(t)\rangle_N. \end{aligned} \quad (5.5)$$

5.2. Classical limit of the future-not-included theory

As we pointed out in refs.,^{16), 27)} eqs.(5.4)(5.5) suggest that the effect of the anti-Hermitian part of the non-Hermitian Hamiltonian \hat{H} disappears in the classical limit, though the theory is defined with \hat{H} at the quantum level. To see this in terms of the expectation value $\langle \hat{\mathcal{O}} \rangle^{AA}$, utilizing eq.(5.5) we give the following expression,

$$\begin{aligned} i\hbar \frac{d}{dt} \langle \hat{\mathcal{O}} \rangle^{AA} &= \langle [\hat{\mathcal{O}}, \hat{H}_h] \rangle^{AA} + \langle F(\hat{\mathcal{O}}, \hat{H}_a) \rangle^{AA}, \\ &\simeq \langle [\hat{\mathcal{O}}, \hat{H}_h] \rangle^{A(t)A(t)}, \end{aligned} \quad (5.6)$$

where $F(\hat{\mathcal{O}}, \hat{H}_a)(t)$, a quantum fluctuation term given by

$$\begin{aligned} F(\hat{\mathcal{O}}, \hat{H}_a)(t) &= \left\{ \hat{\mathcal{O}}, \hat{H}_a - \langle \hat{H}_a \rangle^{AA} \right\} \\ &= \left\{ \hat{\mathcal{O}} - \langle \hat{\mathcal{O}} \rangle^{AA}, \hat{H}_a \right\}, \end{aligned} \quad (5.7)$$

disappears in the classical limit, so we have used the relation $\langle F(\hat{\mathcal{O}}, \hat{H}_a) \rangle^{AA} \simeq 0$.

Substituting \hat{q}_{new} and \hat{p}_{new} for $\hat{\mathcal{O}}$ in eq.(5.6), we obtain

$$\begin{aligned} \frac{d}{dt} \langle \hat{q}_{new} \rangle^{AA} &\simeq \frac{1}{i\hbar} \langle [\hat{q}_{new}, \hat{H}_h] \rangle^{AA} \\ &\simeq \frac{1}{m_{\text{eff}}} \langle \hat{p}_{new} \rangle^{AA}, \end{aligned} \quad (5.8)$$

$$\begin{aligned} \frac{d}{dt} \langle \hat{p}_{new} \rangle^{AA} &\simeq \langle [\hat{p}_{new}, \hat{H}_h] \rangle^{AA} \\ &\simeq -\langle V'_R(\hat{q}_{new}) \rangle^{AA}, \end{aligned} \quad (5.9)$$

where in the last line of each relation m_{eff} and V_R are given by

$$m_{\text{eff}} \equiv m_R + \frac{m_I^2}{m_R} \quad (5.10)$$

and eq.(3.2), and we have taken into account the theorem 1 given in subsection 2.4 and used the approximation that $\hat{q}_{new} \simeq \hat{q}$ and $\hat{p}_{new} \simeq \hat{p}$. Since eq.(5.8) suggests the following momentum relation,

$$p = m_{\text{eff}}\dot{q}, \quad (5.11)$$

we claim that this is the momentum relation in the future-not-included theory. Eq.(5.11) is different from eq.(3.10), which is confirmed to be the momentum relation in the future-included theory. But, in the future-not-included theory, where both $\langle \hat{q}_{new} \rangle^{AA}$ and $\langle \hat{p}_{new} \rangle^{AA}$ are real for \hat{q}_{new} and \hat{p}_{new} replaced with Hermitian \hat{q} and \hat{p} respectively, eq.(3.10) is inconsistent because m is complex. On the other hand, we do not encounter such a contradiction for eq.(5.11) in the future-not-included theory, because m_{eff} is real. Therefore we conclude that the momentum relations in the future-included and future-not-included theories are given by eqs.(3.10)(5.11), respectively. Then, one may have a question why the method of ref.²⁸⁾ to derive eq.(3.10), which was explained in section 3, does not work in the future-not-included theory. Later in section 6 we will come back to this point and explain that the method works even in the future-not-included theory and provides eq.(5.11), if it is properly applied to the future-not-included theory.

Combining eq.(5.8) with eq.(5.9), we obtain the Ehrenfest's theorem,

$$m_{\text{eff}} \frac{d^2}{dt^2} \langle \hat{q}_{new} \rangle^{AA} \simeq -\langle V'_R(\hat{q}_{new}) \rangle^{AA}, \quad (5.12)$$

which suggests that the classical theory of the future-not-included theory is described by not a full action S but S_{eff} defined by

$$S_{\text{eff}} \equiv \int_{T_A}^t dt L_{\text{eff}}, \quad (5.13)$$

$$L_{\text{eff}}(\dot{q}, q) \equiv \frac{1}{2} m_{\text{eff}} \dot{q}^2 - V_R(q). \quad (5.14)$$

Here we note that L_{eff} is different from L_R given in eq.(3.4). Thus we claim that the classical theory of the future-not-included theory is described by $\delta S_{\text{eff}} = 0$. Then the momentum relation given in eq.(5.11) is rewritten as $p = \frac{\partial L_{\text{eff}}}{\partial \dot{q}}$. This is quite in contrast to the classical theory of the future-included theory, which would be described by $\delta S = 0$, where $S = \int_{T_A}^{T_B} dt L$, and the momentum relation is given by eq.(3.10). In addition, the classical Hamiltonian in the future-not-included theory is given by

$$H_R \equiv \text{Re}_q H = \frac{1}{2m_{\text{eff}}} p^2 + V_R(q), \quad (5.15)$$

where H_R is the q -real part of the classical Hamiltonian $H \equiv \frac{1}{2m} p^2 + V(q)$, which is given by replacing \hat{q}_{new} and \hat{p}_{new} with q and p respectively in \hat{H} . In refs.^{16), 27)} introducing a proper

inner product so that the eigenstates of \hat{H} are orthogonal to each other and considering a long time development, we obtained a Hermitian Hamiltonian. But now without using the automatic hermiticity mechanism we have obtained a real Hamiltonian in the classical limit. This is an intriguing property of the future-not-included theory, though restricted to the classical limit. We make a comparison between the future-included and future-not-included theories in table I.

Table I. Comparison between the future-included and future-not-included theories

	future-included theory	future-not-included theory
action	$S = \int_{T_A}^{T_B} dt L$	$S = \int_{T_A}^t dt L$
“expectation value”	$\langle \hat{\mathcal{O}} \rangle^{BA} = \frac{\langle B(t) \hat{\mathcal{O}} A(t) \rangle}{\langle B(t) A(t) \rangle}$	$\langle \hat{\mathcal{O}} \rangle^{AA} = \frac{\langle A(t) \hat{\mathcal{O}} A(t) \rangle}{\langle A(t) A(t) \rangle}$
time development	$i\hbar \frac{d}{dt} \langle \hat{\mathcal{O}} \rangle^{BA}$ $= \langle [\hat{\mathcal{O}}, \hat{H}] \rangle^{BA}$	$i\hbar \frac{d}{dt} \langle \hat{\mathcal{O}} \rangle^{AA}$ $= \langle [\hat{\mathcal{O}}, \hat{H}_h] \rangle^{AA} + \langle \{ \hat{\mathcal{O}} - \langle \hat{\mathcal{O}} \rangle^{AA}, \hat{H}_a \} \rangle^{AA}$ $\simeq \langle [\hat{\mathcal{O}}, \hat{H}_h] \rangle^{AA}$
classical theory	$\delta S = 0$	$\delta S_{\text{eff}} = 0, S_{\text{eff}} = \int_{T_A}^t dt L_{\text{eff}}$
momentum relation	$p = m\dot{q}$	$p = m_{\text{eff}}\dot{q}$

5.3. Another method to see the time development of $\langle \hat{\mathcal{O}} \rangle^{AA}$ by re-choosing the B state

The quantity $\langle \hat{\mathcal{O}} \rangle^{BA}$ in the future-included theory behaves as an expectation value though its looking is a matrix element, and it time-develops according to a very simple expression of eq.(4.9). On the other hand, the expectation value $\langle \hat{\mathcal{O}} \rangle^{AA}$ in the future-not-included theory time-develops more complicatedly at the quantum level with the additional term $\langle \{ \hat{\mathcal{O}}, \hat{H}_a - \langle \hat{H}_a \rangle \} \rangle^{AA}$, as seen in eq.(5.6). Hence we are motivated to study whether we can simplify the description of the time development of $\langle \hat{\mathcal{O}} \rangle^{AA}$ by rewriting it formally in the expression of the future-included theory and utilizing the simple time development of the future-included theory. Even if we cannot make it simpler, it would be interesting to reproduce and understand the time development of the future-not-included theory from a different point of view via the future-included theory. At least this would become a consistency check of the theory, and we could claim that the future-included theory can be used as a mathematical tool to compute the time development of $\langle \hat{\mathcal{O}} \rangle^{AA}$. Therefore, in this subsection, we try to describe the time development of the expectation value of the future-not-included theory $\langle \hat{\mathcal{O}} \rangle^{AA}$ by making use of the future-included theory.

We begin with putting the condition

$$\langle q | B(t) \rangle = \langle q | A(t) \rangle \quad (5.16)$$

on the B state at some time t .*) We call it “re-choosing” of the B state. Expressing the B state re-chosen at t as $|B_t(t')\rangle$, where t' is a formal time to allow the time-development as a B state, we have the following relation for the time t ,

$$|B_t(t)\rangle = |A(t)\rangle. \quad (5.17)$$

Then eq.(4.6) is rewritten as

$$\langle \hat{\mathcal{O}} \rangle^{AA} = \frac{\langle B_t(t) | \hat{\mathcal{O}} | A(t) \rangle}{\langle B_t(t) | A(t) \rangle} \equiv \langle \hat{\mathcal{O}} \rangle^{B_t A} \quad (5.18)$$

for each t . In a realistic future-included theory it would be a very strange accident if we have the relation of eq.(5.16) even at one time. Hence the re-choosing cannot be taken seriously. We just look for some formal rule to use the future-included theory as long as possible but to obtain the future-not-included theory as our result.

The re-chosen B state $|B_t(t')\rangle$ obeys

$$i\hbar \frac{d}{dt} |B_t(t)\rangle = H |B_t(t)\rangle, \quad (5.19)$$

$$i\hbar \frac{\partial}{\partial t'} |B_t(t')\rangle = H^\dagger |B_t(t')\rangle, \quad (5.20)$$

which come from eqs.(4.3)(4.4) respectively. Using eqs.(5.17)(5.20) we can calculate the time derivative of $|A(t)\rangle$ as

$$\frac{d}{dt} |A(t)\rangle = \left(\frac{\partial}{\partial t} |B_t(t')\rangle \right) |_{t'=t} - \frac{i}{\hbar} H^\dagger |B_t(t)\rangle. \quad (5.21)$$

Since eq.(5.20) provides the expression

$$|B_t(t')\rangle = e^{-\frac{i}{\hbar} H^\dagger (t' - t'')} |B_t(t'')\rangle = e^{-\frac{i}{\hbar} H^\dagger (t' - t)} |B_t(t)\rangle, \quad (5.22)$$

we obtain

$$\frac{\partial}{\partial t} |B_t(t')\rangle = \frac{i}{\hbar} \left(H^\dagger |B_t(t')\rangle - e^{-\frac{i}{\hbar} H^\dagger (t' - t)} H |B_t(t)\rangle \right). \quad (5.23)$$

For $t' = t$ this is expressed as

$$\begin{aligned} i\hbar \left(\frac{\partial}{\partial t} |B_t(t')\rangle \right) |_{t'=t} &= (H - H^\dagger) |B_t(t')\rangle |_{t'=t} \\ &= 2H_a e^{-\frac{i}{\hbar} H^\dagger (t - t'')} |B_t(t'')\rangle, \end{aligned} \quad (5.24)$$

*) We cannot simply put eq.(5.16) except for at one value of t , because the states $|A(t)\rangle$ and $|B(t)\rangle$ time-develop differently according to eqs.(4.3)(4.4) respectively.

where the left-hand side is rewritten as

$$i\hbar \left\{ \frac{\partial}{\partial t} \left(e^{-iH^\dagger(t-t'')} |B_t(t'')\rangle \right) \right\} |_{t'=t} = e^{-\frac{i}{\hbar}H^\dagger(t-t'')} i\hbar \frac{\partial}{\partial t} |B_t(t'')\rangle. \quad (5.25)$$

Therefore, we obtain

$$i\hbar \frac{\partial}{\partial t} |B_t(t'')\rangle = \mathcal{U}_{t'',t}^{-1} 2H_a \mathcal{U}_{t'',t} |B_t(t'')\rangle, \quad (5.26)$$

where we have introduced

$$\mathcal{U}_{t'',t} = e^{-\frac{i}{\hbar}H^\dagger(t-t'')}. \quad (5.27)$$

Next we calculate the time derivative of $\langle \hat{\mathcal{O}} \rangle^{AA}$,

$$\frac{\partial}{\partial t} \langle \hat{\mathcal{O}} \rangle^{A(t)A(t)} = \left\{ \frac{\partial}{\partial t} \langle \hat{\mathcal{O}} \rangle^{B_t(t')A(t')} \right\} |_{t'=t} + \left\{ \frac{\partial}{\partial t'} \langle \hat{\mathcal{O}} \rangle^{B_t(t')A(t')} \right\} |_{t'=t}, \quad (5.28)$$

where $\langle \hat{\mathcal{O}} \rangle^{B_t(t')A(t')} = \frac{\langle B_t(t') | \hat{\mathcal{O}} | A(t') \rangle}{\langle B_t(t') | A(t') \rangle}$ is formally a good classical solution in the future-included theory for each t' as long as equation of motion is considered. Indeed the second term of eq.(5.28) is expressed as

$$\left\{ \frac{\partial}{\partial t'} \langle \hat{\mathcal{O}} \rangle^{B_t(t')A(t')} \right\} |_{t'=t} = \frac{i}{\hbar} \langle [H, \hat{\mathcal{O}}] \rangle^{B_t(t')A(t')} |_{t'=t}. \quad (5.29)$$

On the other hand, the first term of eq.(5.28) does not become a simple expression. We can rewrite this by utilizing eq.(5.26) as follows,

$$\begin{aligned} \left\{ \frac{\partial}{\partial t} \langle \hat{\mathcal{O}} \rangle^{B_t(t')A(t')} \right\} |_{t'=t} &= \left\{ -\frac{2i}{\hbar} \frac{1}{\langle B_t(t') | A(t') \rangle} \langle B_t(t') | U_{t',t}^\dagger H_a (U_{t',t}^{-1})^\dagger \hat{\mathcal{O}} | A(t') \rangle \right. \\ &\quad \left. + \frac{2i}{\hbar} \frac{\langle B_t(t') | \hat{\mathcal{O}} | A(t') \rangle}{(\langle B_t(t') | A(t') \rangle)^2} \langle B_t(t') | U_{t',t}^\dagger H_a (U_{t',t}^{-1})^\dagger | A(t') \rangle \right\} |_{t'=t} \\ &= \frac{2i}{\hbar} \left\{ \langle \hat{\mathcal{O}} \rangle^{A(t)A(t)} \langle H_a \rangle^{A(t)A(t)} - \langle H_a \hat{\mathcal{O}} \rangle^{A(t)A(t)} \right\}, \\ &= \frac{1}{i\hbar} \left[-\langle [\hat{\mathcal{O}}, H_a] \rangle^{A(t)A(t)} + \langle \left\{ \hat{\mathcal{O}} - \langle \hat{\mathcal{O}} \rangle^{A(t)A(t)}, H_a \right\} \rangle^{A(t)A(t)} \right]. \end{aligned} \quad (5.30)$$

Substituting eqs.(5.29)(5.30) for eq.(5.28), we obtain eq.(5.6). Thus we have shown that we can derive the time development of $\langle \hat{\mathcal{O}} \rangle^{AA}$ the expectation value in the future-not-included theory by making use of the future-included theory. Especially we have explicitly seen that it is the first term of eq.(5.28) that provides the anti-commutator term, which disappears in the classical limit, besides the commutator $\langle [\hat{\mathcal{O}}, H_a] \rangle^{A(t)A(t)}$. As a result, this method is not so simple, but it would be interesting in the sense that this provides another way to understand the time development of the future-not-included theory. Indeed we have seen that the time development of $\langle \hat{\mathcal{O}} \rangle^{AA}$ is expressed as the simple time development of $\langle \hat{\mathcal{O}} \rangle^{BA}$ and slightly complicated correction due to the formal re-choosing of the B state.

§6. Reconsideration of the method to derive the momentum relation via Feynman path integral in the future-not-included theory

In the foregoing sections we have seen that the momentum relation of eq.(3·10) derived via FPI in ref.²⁸⁾ is valid in the future-included theory, because it is consistent with eq.(4·10), which was derived by looking at the time development of $\langle \hat{q}_{new} \rangle^{BA}$ in the future-included theory. In eq.(5·11) we obtained another momentum relation in the future-not-included theory by analyzing the time development of $\langle \hat{q}_{new} \rangle^{AA}$. Now one might have a question why the method of ref.²⁸⁾ to derive the momentum relation via FPI, which was reviewed in section 3, is not valid in the future-not-included theory. The reason is as follows: In the method of ref.²⁸⁾ we analyzed the time development of ξ -parametrized state in a transition amplitude from the initial time t_i to the final time t_f , where the present time t is supposed to be between t_i and t_f . Such a transition amplitude is similar to that in the future-included theory, which is written as

$$\langle B(t)|A(t) \rangle = \langle B(T_B)| \exp \left(-\frac{i}{\hbar} \hat{H}(T_B - T_A) \right) |A(T_A) \rangle, \quad (6.1)$$

where the present time t is between T_A and T_B . On the other hand, in the future-not-included theory the transition amplitude is given by

$$\langle A(t)|A(t) \rangle = \langle A(T_A)| \exp \left(\frac{i}{\hbar} \hat{H}^\dagger(t - T_A) \right) \exp \left(-\frac{i}{\hbar} \hat{H}(t - T_A) \right) |A(T_A) \rangle, \quad (6.2)$$

so we have to consider a path starting from the initial time T_A to the present time t , and also that going backward from t to T_A . In this section we discuss how to apply the method of ref.²⁸⁾ to derive the momentum relation via Feynman path integral, which was reviewed in section 3, to the future-not-included theory.

6.1. Formal Lagrangian in the future-not-included theory

To apply the method of ref.²⁸⁾ to the future-not-included theory, we formally rewrite the transition amplitude $\langle A(t)|A(t) \rangle$ into another expression similar to $\langle B(t)|A(t) \rangle$, and introduce a formal Lagrangian L_{formal} . We argue that using this formal Lagrangian L_{formal} on behalf of the original Lagrangian L in the method of ref.²⁸⁾ we obtain the momentum relation in the future-not-included theory, eq.(5·11).

In the future-not-included theory we can rewrite eq.(6·2) as the following path integral,

$$I \equiv \int_C \mathcal{D}q \int_{C'} \mathcal{D}q' e^{-\frac{i}{\hbar} S_{T_A \text{ to } t(q)^{*q}}} e^{\frac{i}{\hbar} S_{T_A \text{ to } t(q')}} \psi_A(q_{T_A}, T_A)^{*q_{T_A}} \psi_A(q'_{T_A}, T_A). \quad (6.3)$$

At an intermediate time t' such that $T_A < t' < t$ we would be allowed to use a kind of future-included formulation, because it looks as if there were a future for t' . But for the present time

t there is no future but only the past, so we have to be careful about quantities at the time t , especially \dot{q} etc. Therefore, we transform I into the expression like a transition amplitude from the time T_A to T_B by inverting the time direction of the transition amplitude from T_A to t so that t becomes an intermediate time. For this purpose we express $S_{T_A \text{ to } t}(q)^{*q}$ as

$$\begin{aligned} S_{T_A \text{ to } t}(q)^{*q} &= \int_{T_A}^t dt' L(q(t'), \dot{q}(t'))^{*q} \\ &= \int_t^{-T_A+2t} dt'' L(q_{\text{formal}}(t'', t), -\partial_{t''} q_{\text{formal}}(t'', t))^{*q_{\text{formal}}}, \end{aligned} \quad (6.4)$$

where in the second equality we have changed the variable by

$$t'' = -t' + 2t, \quad (6.5)$$

and introduced the formal coordinate q_{formal} by

$$q_{\text{formal}}(t'', t) \equiv q(-t'' + 2t) = q(t'), \quad (6.6)$$

which has the time dependence of not only t but also t'' and suggests

$$q(t) = q_{\text{formal}}(t, t). \quad (6.7)$$

Then I is written as

$$\begin{aligned} I &= \int_{C'} \mathcal{D}q' \int_{C''} \mathcal{D}q_{\text{formal}} \exp \left[\frac{i}{\hbar} \int_{T_A}^t dt' L(q'(t'), \dot{q}'(t')) \right] \\ &\quad \times \exp \left[-\frac{i}{\hbar} \int_t^{T_B} dt'' L(q_{\text{formal}}(t'', t), -\partial_{t''} q_{\text{formal}}(t'', t))^{*q_{\text{formal}}} \right] J \psi_A(q'_{T_A}, T_A), \end{aligned} \quad (6.8)$$

where C'' is a contour of $q_{\text{formal}}(t'', t)$, which is obtained by a reflection of C at t in the time direction, and J is given by

$$\begin{aligned} J &= \int_{C'''} \mathcal{D}q'_{\text{formal}} \exp \left[-\frac{i}{\hbar} \int_{T_B}^{-T_A+2t} dt'' L(q'_{\text{formal}}(t'', t), -\partial_{t''} q'_{\text{formal}}(t'', t))^{*q'_{\text{formal}}} \right] \\ &\quad \times \psi_A(q'_{\text{formal}}(-T_A + 2t, t), T_A)^{*q'_{\text{formal}}} \\ &= \left\{ \int_{C'''} dq'_{\text{formal}}(-T_A + 2t, t) {}_m \langle q'_{\text{formal}}(-T_A + 2t, t) | e^{-\frac{i}{\hbar} \hat{H}(-T_A+2t-T_B)} | q'_{\text{formal}}(T_B, t) \rangle \right. \\ &\quad \left. \times {}_m \langle q'_{\text{formal}}(-T_A + 2t, t) | A(T_A) \rangle \right\}^{*q'_{\text{formal}}}. \end{aligned} \quad (6.9)$$

Using the relation

$$\begin{aligned} &{}_m \langle q'_{\text{formal}}(-T_A + 2t, t) | e^{-\frac{i}{\hbar} \hat{H}(-T_A+2t-T_B)} | q'_{\text{formal}}(T_B, t) \rangle \\ &= {}_m \langle q'_{\text{formal}}(T_B, t) | e^{-\frac{i}{\hbar} \hat{H}(-T_A+2t-T_B)} | q'_{\text{formal}}(-T_A + 2t, t) \rangle, \end{aligned} \quad (6.10)$$

we obtain a simple expression for J ,

$$\begin{aligned} J &= \langle A(2t - T_B) | q'_{\text{formal}}(T_B, t) \rangle \\ &= \psi_A(q'_{\text{formal}}(T_B, t), 2t - T_B)^{*q'_{\text{formal}}}. \end{aligned} \quad (6.11)$$

We note that the time $2t - T_B$ is not so far from T_A because we suppose $T_B \simeq -T_A \simeq \infty$. Expressing $q'(t')$ for $T_A \leq t' \leq t$ as $q_{\text{formal}}(t', t)$ formally, we can rewrite the integral I as

$$\begin{aligned} I &\simeq \int \mathcal{D}q_{\text{formal}} \exp \left[\frac{i}{\hbar} \int_{T_A}^{T_B} dt' \{-\epsilon(t' - t)\} L_{\text{formal}}(q_{\text{formal}}(t', t), \partial_{t'} q_{\text{formal}}(t', t), t' - t) \right] \\ &\quad \times \psi_A(q_{\text{formal}}(T_B, t), 2t - T_B)^{*q_{\text{formal}}} \psi_A(q_{\text{formal}}(T_A, t), T_A), \end{aligned} \quad (6.12)$$

where $\epsilon(t)$ is a step function defined as 1 for $t > 0$ and -1 for $t < 0$, and we have introduced the formal Lagrangian L_{formal} by

$$\begin{aligned} &L_{\text{formal}}(q_{\text{formal}}(t', t), \partial_{t'} q_{\text{formal}}(t', t), t' - t) \\ &\equiv \text{Re}_{q_{\text{formal}}} L(q_{\text{formal}}(t', t), -\epsilon(t' - t) \partial_{t'} q_{\text{formal}}(t', t)) \\ &\quad - i\epsilon(t' - t) \text{Im}_{q_{\text{formal}}} L(q_{\text{formal}}(t', t), -\epsilon(t' - t) \partial_{t'} q_{\text{formal}}(t', t)) \\ &= \frac{1}{2} m_{\text{formal}}(t' - t) (\partial_{t'} q_{\text{formal}}(t', t))^2 - V_{\text{formal}}(q_{\text{formal}}(t', t), t' - t). \end{aligned} \quad (6.13)$$

Here $m_{\text{formal}}(t' - t)$ and $V_{\text{formal}}(q_{\text{formal}}(t', t), t' - t)$ are formal mass and potential given by

$$m_{\text{formal}}(t' - t) \equiv m_R - i\epsilon(t' - t)m_I, \quad (6.14)$$

$$V_{\text{formal}}(q_{\text{formal}}(t', t), t' - t) \equiv V_R(q_{\text{formal}}(t', t)) - i\epsilon(t' - t)V_I(q_{\text{formal}}(t', t)). \quad (6.15)$$

In eq.(6.12) we have defined L_{formal} by extracting the factor $-\epsilon(t' - t)$, which is caused by the time reflection of eq.(6.5). L_{formal} looks like a non-translational invariant Lagrangian depending on t' , and t is just a selected point in time. Therefore, we normally have to think of t' as the time in using L_{formal} .

One may think that the transition amplitude of eq.(6.2) can be expressed as

$$\begin{aligned} &\langle A(t) | A(t) \rangle \\ &= \langle A(T_B) | e^{\frac{i}{\hbar} H^\dagger(t - T_B)} e^{-\frac{i}{\hbar} H(t - T_A)} | A(T_A) \rangle \\ &= \int \mathcal{D}q \mathcal{D}q' \psi_A(q_{T_B}, T_B)^{*q} e^{\frac{i}{\hbar} S_{t \text{ to } T_B}(q)^{*q}} e^{\frac{i}{\hbar} S_{T_A \text{ to } t}(q')} \psi_A(q'_{T_A}, T_A) \delta(q_t - q'_t) \\ &= \int \mathcal{D}q \exp \left[\frac{i}{\hbar} \int_{T_A}^{T_B} dt' \{ \theta(t - t') L(q) + \theta(t' - t) L(q)^{*q} \} \right] \psi_A(q_{T_B}, T_B)^{*q} \psi_A(q_{T_A}, T_A), \\ &= \int \mathcal{D}q \exp \left[\frac{i}{\hbar} \int_{T_A}^{T_B} dt' L_{\text{formal}, 2} \right] \psi_A(q_{T_B}, T_B)^{*q} \psi_A(q_{T_A}, T_A), \end{aligned} \quad (6.16)$$

where $\theta(t) = \frac{1}{2}(\epsilon(t) + 1)$ is a step function defined as 1 for $t > 0$ and 0 for $t < 0$, and $L_{\text{formal}, 2}$ is given by

$$L_{\text{formal}, 2}(q(t'), \dot{q}(t'), t' - t) \equiv \text{Re}_q L(q(t'), \dot{q}(t')) - i\epsilon(t' - t)\text{Im}_q L(q(t'), \dot{q}(t')). \quad (6.17)$$

We might think that this rewriting is also good for our purpose, but it is not the case, since in eq.(6.16) only the half of the original path, i.e. the path going from T_A to t , is mapped onto the time interval $[T_A, T_B]$ over which $L_{\text{formal}, 2}$ is time-integrated.

6.2. Momentum relation in the future-not-included theory

Since we have found the formal Lagrangian L_{formal} , we try to obtain the momentum relation in the future-not-included theory by replacing L with L_{formal} in the method of ref.²⁸⁾ Then we obtain the formal momentum $p_{\text{formal}}(t', t)$,

$$\begin{aligned} p_{\text{formal}}(t', t) &= \frac{\partial L_{\text{formal}}(q_{\text{formal}}(t', t), \partial_{t'} q_{\text{formal}}(t', t), t' - t)}{\partial(\partial_{t'} q_{\text{formal}}(t', t))} \\ &= m_{\text{formal}}(t' - t) \partial_{t'} q_{\text{formal}}(t', t). \end{aligned} \quad (6.18)$$

Since $\partial_{t'} q_{\text{formal}}$ could jump up around $t' = t$, we slightly take the time average of this around $t' = t$ expecting a finite observation time. Thus the time derivative of $q(t)$, which is given in eq.(6.7), is evaluated as

$$\begin{aligned} \frac{d}{dt} q(t) &\simeq \left\{ \frac{\partial}{\partial t'} q_{\text{formal}}(t', t) \right\} \Big|_{t'=t} \\ &\simeq \frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} dt' \partial_{t'} q_{\text{formal}}(t', t) \\ &= \frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} dt' \frac{p_{\text{formal}}(t', t)}{m_{\text{formal}}(t' - t)} \\ &\simeq \frac{1}{m_{\text{eff}}} p(t), \end{aligned} \quad (6.19)$$

where in the first equality we have used the relation

$$\left\{ \frac{\partial}{\partial t'} q_{\text{formal}}(t', t) \right\} \Big|_{t'=t} = 0, \quad (6.20)$$

which holds because $q_{\text{formal}}(t', t)$ is independent of t for $t' < t$ and supposed to be smooth. In the second equality we have changed the expression into the time average of $\partial_{t'} q_{\text{formal}}$ around $t' = t$. In the third and fourth equalities we have used eq.(6.18), and supposed that p_{formal} changes very little near $t' = t$, and m_{eff} and $p(t)$ are given by eq.(5.10) and

$$p(t) \equiv p_{\text{formal}}(t, t). \quad (6.21)$$

Thus we have succeeded in reproducing eq.(5.11), the momentum relation in the future-not-included theory, by utilizing the method of ref.²⁸⁾ Eq.(6.19) is consistent with eq.(5.8), which was derived by analyzing the time derivative of $\langle \hat{q}_{new} \rangle^{AA}$.

Finally we make a couple of remarks. If we naively average p_{formal} first, then we might expect the relation like $p = m_R \dot{q}$, which is not right. It is $\partial_{t'} q_{\text{formal}}$, not p_{formal} , that we should average because the former includes the derivative with regard to t' , which could jump up around $t' = t$. Similarly, it is not reasonable to take the time average of L_{formal} because it includes $\partial_{t'} q_{\text{formal}}(t', t)$, which we need to average separately.

§7. Discussion

In this paper, after reviewing the complex coordinate formalism,²⁷⁾ the method to derive the momentum relation via Feynman path integral (FPI)²⁸⁾ and some properties of the future-included theory studied in ref.,³⁴⁾ we provided the momentum relation and classical limit in the future-not-included theory, which are different from those in the future-included theory. In section 2 we reviewed the complex coordinate formalism,²⁷⁾ which is a kind of generalized bra-ket formalism so that we can properly deal with complex coordinate q and momentum p . In section 3 following ref.²⁸⁾ we reviewed the method to derive the momentum relation by analyzing the time development of ξ -parametrized state via FPI, and obtained the momentum relation $p = m\dot{q}$. In section 4 based on ref.³⁴⁾ we saw that the quantity $\langle \hat{\mathcal{O}} \rangle^{BA}$ behaves as if an expectation value of some operator $\hat{\mathcal{O}}$ in the future-included theory, and derived the momentum relation $\langle \hat{p}_{new} \rangle^{BA} = m \frac{d}{dt} \langle \hat{q}_{new} \rangle^{BA}$, which is consistent with that given in the previous section.

In section 5 we studied the future-not-included theory and saw that the expectation value $\langle \hat{\mathcal{O}} \rangle^{AA}$ time-develops not so cleanly compared to $\langle \hat{\mathcal{O}} \rangle^{BA}$ because of the presence of an additional anti-commutator term. But this anti-commutator term is a quantum fluctuation term, so it vanishes in the classical limit. Thus we obtained the relation $\langle \hat{p}_{new} \rangle^{AA} = m_{\text{eff}} \frac{d}{dt} \langle \hat{q}_{new} \rangle^{AA}$ and claimed that $p = m_{\text{eff}} \dot{q}$ is the momentum relation in the future-not-included theory. Moreover, we argued that classical theory of the future-not-included theory is described by not a full action S but a certain real action S_{eff} , which is not the real part of S . This is quite in contrast to the future-included theory, whose classical theory is described by a full action S . Furthermore, in subsection 5.3 we offered another way to understand the time development of the future-not-included theory via the future-included theory. The above studies suggest that the method of ref.²⁸⁾ to derive the momentum relation via FPI is valid in the future-included theory, but not in the future-not-included theory. In ref.²⁸⁾ we derived the momentum relation $p = m\dot{q}$ by considering a transition amplitude from some initial time

to final time, which is similar to the transition amplitude in the future-included theory, but not to that in the future-not-included theory. In section 6 we provided a way to apply the method of ref.²⁸⁾ properly to the future-not-included theory by rewriting the transition amplitude in the future-not-included theory into another expression similar to the transition amplitude in the future-included theory and introducing a formal Lagrangian. Indeed we explicitly showed that we can derive the momentum relation $p = m_{\text{eff}}\dot{q}$ in the future-not-included theory via this method. We summarized the difference between the future-included and future-not-included theories in table I.

Finally let us seek the possibility to define some sensible formal Hamiltonian in the future-not-included theory starting from the formal Lagrangian L_{formal} in the method of ref.,²⁸⁾ where we derived not only momentum relation but also Hamiltonian via path integral. In section 6 we provided a way to utilize the method by introducing L_{formal} . So if we use L_{formal} , we would obtain a formal Hamiltonian. Replacing L with L_{formal} results in replacing m and V with $m_{\text{formal}}(t' - t)$ and $V_{\text{formal}}(q(t'), t' - t)$ respectively in the expression of eq.(3.12). Thus we would obtain a formal Hamiltonian, $\hat{H}_{\text{formal}}(t' - t) = \frac{1}{2m_{\text{formal}}(t' - t)}(\hat{p}_{\text{new}})^2 + V_{\text{formal}}(\hat{q}_{\text{new}}, t' - t)$, where $m_{\text{formal}}(t' - t)$ and $V_{\text{formal}}(q(t'), t' - t)$ are given in eqs.(6.14)(6.15) respectively. But this Hamiltonian does not have a good physical meaning. We can see this by trying to introduce some formal state $|\psi(t', t)\rangle_{\text{formal}}$, which time-develops according to the following formal Schrödinger equation $i\hbar \frac{\partial}{\partial t'} |\psi(t', t)\rangle_{\text{formal}} = \hat{H}_{\text{formal}}(t' - t) |\psi(t', t)\rangle_{\text{formal}}$ with an initial condition $|\psi(T_A, t)\rangle_{\text{formal}} = |A(T_A)\rangle$. Let us define some effective state $|\psi(t)\rangle_{\text{eff}}$ by $|\psi(t)\rangle_{\text{eff}} = |\psi(t, t)\rangle_{\text{formal}}$, where $|\psi(t', t)\rangle_{\text{formal}}$ is supposed to be smooth in t' . Then taking the time average of the formal Schrödinger equation, we would obtain the following effective Schrödinger equation, $i\hbar \left(\frac{\partial}{\partial t} |\psi(t)\rangle_{\text{eff}} - \frac{\partial}{\partial t} |\psi(t', t)\rangle_{\text{formal}}|_{t'=t} \right) = \hat{H}_{\text{eff}} |\psi(t)\rangle_{\text{eff}}$, where $|\psi(t)\rangle_{\text{eff}}$ obeys an initial condition $|\psi(T_A)\rangle_{\text{eff}} = |A(T_A)\rangle$, and \hat{H}_{eff} is expressed as $\hat{H}_{\text{eff}} \equiv \frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} dt' \hat{H}_{\text{formal}}(t' - t) = \frac{1}{2m_{\text{eff}}}(\hat{p}_{\text{new}})^2 + V_R(\hat{q}_{\text{new}}) \simeq \hat{H}_h$. Here in the last equality expecting that this \hat{H}_{eff} is put in some matrix elements we have used the approximation $\hat{q}_{\text{new}}^\dagger \simeq \hat{q}_{\text{new}}$ and $\hat{p}_{\text{new}}^\dagger \simeq \hat{p}_{\text{new}}$ based on the theorem 1 given in subsection 2.4. The effective Schrödinger equation shows that \hat{H}_{eff} is not a Hamiltonian because we have the second term on the left-hand side, though the time average of \hat{H}_{eff} becomes the classical Hamiltonian in the future-not-included theory H_R , which is given in eq.(5.15). Therefore \hat{H}_{formal} is not a sensible Hamiltonian. It would be interesting if we could find some sensible formal Hamiltonian in the future, but practically we do not need this, because we know that the quantum Hamiltonian of the future-not-included theory is \hat{H} by definition, and also we found in this paper that the classical theory is described by H_R .

Now that we have understood the general classical properties of the future-not-included complex action theory, it would be desirable to study the dynamics of the theory in some con-

crete model. We will work on both of the future-included and future-not-included theories, and report some progress in the future.

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References

- 1) H. B. Nielsen and M. Ninomiya, the proceedings of Bled 2006 -What Comes Beyond the Standard Models-, p.87-124, arXiv:hep-ph/0612250.
- 2) H. B. Nielsen and M. Ninomiya, arXiv:0802.2991 [physics.gen-ph]; Int. J. Mod. Phys. A **23** (2008) 919; Prog. Theor. Phys. **116** (2007) 851.
- 3) H. B. Nielsen and M. Ninomiya, the proceedings of Bled 2007 -What Comes Beyond the Standard Models- , p.144-185(arXiv:0711.3080 [hep-ph]).
- 4) H. B. Nielsen and M. Ninomiya, arXiv:0910.0359 [hep-ph].
- 5) H. B. Nielsen, in Proceedings of the conference QTFR-5, June 14-18, 2009. Vaxjo, Sweden, arXiv:0911.4005 [quant-ph].
- 6) H. B. Nielsen and M. Ninomiya, the proceedings of Bled 2010 -What Comes Beyond the Standard Models-, p.138-157(arXiv:1008.0464 [physics.gen-ph]).
- 7) H. B. Nielsen, arXiv:1006.2455 [physic.gen-ph].
- 8) H. B. Nielsen and M. Ninomiya, arXiv:hep-th/0701018.
- 9) H. B. Nielsen, arXiv:0911.3859 [gr-qc].
- 10) H. B. Nielsen, M. S. Mankoc Borstnik, K. Nagao and G. Moultaka, the proceedings of Bled 2010 -What Comes Beyond the Standard Models-, p.211-216(arXiv:1012.0224 [hep-ph]).
- 11) S. Garcia, Z. Guralnik and G. S. Guralnik, arXiv:hep-th/9612079.
- 12) G. Guralnik and Z. Guralnik, Annals Phys. **325**, 2486 (2010).
- 13) C. Pehlevan and G. Guralnik, Nucl. Phys. B **811**, 519 (2009).
- 14) D. D. Ferrante and G. S. Guralnik, arXiv:0809.2778 [hep-th].
- 15) D. D. Ferrante, arXiv:0904.2205 [hep-th].
- 16) K. Nagao and H. B. Nielsen, Prog. Theor. Phys. **125** No. 3, 633 (2011).
- 17) F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. 213 (1992) 74.

- 18) A. Mostafazadeh, J. Math. Phys. **43**, 3944 (2002).
- 19) A. Mostafazadeh, J. Math. Phys. **44**, 974 (2003).
- 20) C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
- 21) C. M. Bender, S. Boettcher and P. Meisinger, J. Math. Phys. **40**, 2201 (1999).
- 22) C. M. Bender, Contemp. Phys. **46**, 277 (2005).
- 23) C. M. Bender, Rept. Prog. Phys. **70**, 947 (2007).
- 24) A. Mostafazadeh, Int. J. Geom. Methods Mod. Phys. **07**, 1191 (2010).
- 25) A. Mostafazadeh, Phys. Scripta **82**, 038110 (2010).
- 26) M. Fukuma, Y. Sakatani and S. Sugishita, arXiv:1301.7352 [hep-th].
- 27) K. Nagao and H. B. Nielsen, Prog. Theor. Phys. **126** No. 6, 1021 (2011) [errata: Prog. Theor. Phys. **127** No. 6, 1131 (2012)].
- 28) K. Nagao and H. B. Nielsen, Int. J. Mod. Phys. A. **27**, 1250076 (2012).
- 29) Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
- 30) Y. Aharonov, S. Popescu, and J. Tollaksen, Physics Today **63**, 27 (2010).
- 31) Y. Aharonov and J. Tollaksen, in *Visions of Discovery: New Light on Physics Cosmology and Consciousness*, edited by R. Y. Chiao, M. L. Cohen, A. J. Leggett, W. D. Phillips, and C. L. Harper, Jr. (Cambridge University Press, 2011), p. 105.
- 32) Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics*, Vol. 1, edited by J. G. Muga, R. Sala Mayato, and I. L. Egusquiza (Springer, 2008) p. 399.
- 33) Y. Shikano, “Measurements in Quantum Mechanics”, edited by M. R. Pahlavani (InTech, 2012) p. 75, arXiv:1110.5055 [quant-ph].
- 34) K. Nagao and H. B. Nielsen, Prog. Theor. Exp. Phys. (2013) 023B04.
- 35) K. Nagao and H. B. Nielsen, Proc. Bled 2012: What Comes Beyond the Standard Models, pp86-93 (2012), arXiv:1211.7269 [quant-ph].
- 36) N. Nakanishi, Prog. Theor. Phys. **19** No. 6, 607 (1958).